Linearisation of overdamped Cosserat rod dynamics

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1 Non-dimensionalisation of dynamics

Our equations of motion are

$$D_t\vec{\theta} = D_u\vec{V} \tag{1}$$
$$\partial_t\vec{\pi} = D_u\vec{\Omega} \tag{2}$$
$$\rho A D_t\vec{V} = D_u\vec{F} - \gamma^T\vec{V} + \vec{f} \tag{3}$$
$$D_t\vec{L} = D_u\vec{M} + \vec{\theta}\times\vec{F} - \gamma^R\vec{\Omega} + \vec{m} \tag{4}$$

We set

$$t = \tau\tilde{t}$$
$$u = L_0\tilde{u}$$
$$\vec{\theta} = \tilde{\vec{\theta}}$$
$$\vec{\pi} = L_0^{-1}\tilde{\vec{\pi}}$$
$$\vec{V} = \frac{L_0}{\tau}\tilde{\vec{V}}$$
$$\vec{\Omega} = \tau^{-1}\tilde{\vec{\Omega}}$$
$$\vec{L} = \frac{\rho A L_0^2}{\tau}\tilde{\vec{L}}$$
$$\vec{F} = F_0\tilde{\vec{F}}$$
$$\vec{M} = M_0\tilde{\vec{M}}$$
$$\vec{f} = \frac{F_0}{L}\tilde{\vec{f}}$$
$$\vec{m} = \frac{M_0}{L_0}\tilde{\vec{m}}$$

where τ, F_0 and M_0 are time, force and moment scales that will be specified later. As $\vec{L} = I\vec{\Omega}$, we also define the dimensionless moment of inerta $\tilde{I} = \frac{1}{\rho A L_0^2} I$. The resulting equations of motion become

$$D_{\tilde{t}} \tilde{\vec{\theta}} = D_{\tilde{u}} \tilde{\vec{V}} \tag{5}$$

$$\partial_{\tilde{t}} \vec{\pi} = D_{\tilde{u}} \tilde{\vec{\Omega}} \tag{6}$$

$$\alpha^T D_{\tilde{t}} \tilde{\vec{V}} = D_{\tilde{u}} \tilde{\vec{F}} - \beta^T \tilde{\vec{V}} + \tilde{\vec{f}} \tag{7}$$

$$\alpha^R D_{\tilde{t}} \tilde{\vec{L}} = D_{\tilde{u}} \tilde{\vec{M}} + \zeta \vec{\theta} \times \tilde{\vec{F}} - \zeta \lambda \tilde{\vec{\Omega}} + \tilde{\vec{m}} \tag{8}$$

where

$$\alpha^T = \frac{\rho A L_0^2/\tau^2}{F_0}$$
$$\beta^T = \frac{(L_0/\tau)\gamma^T}{F_0/L_0}$$
$$\alpha^R = \frac{\rho A L_0^3/\tau^2}{M_0}$$
$$\zeta = \frac{F_0}{M_0/L_0}$$
$$\lambda = \frac{\gamma^R}{\gamma^T L_0^2}$$

α^T can be seen as the ratio of the characeristic inertial forces compared to the characteristic internal force amplitude F_0. β^T is the ratio of the characteristic frictional force to the internal force. α^R is the ratio of the characteristic inertial moment to the characteristic moment amplitude M_0. ζ compares the characteristic force amplitude to the moment. λ compares the translation to the rotational friction.

Without loss of generality we now set $\beta^T = 1$, which means $\tau = \frac{\gamma^T L_0^2}{F_0}$ which is the characteristic damping time-scale for a simple harmonic oscillator. We also set $\zeta = 1$, which ensures that if $\tilde{\vec{F}}$ and $\tilde{\vec{M}}$ are of the same order, then $\vec{F}$ and $\vec{M}/L_0$ are as well. We also then have that $\alpha^R = \alpha^T = \alpha = \frac{\rho A L_0^2/\tau^2}{F_0}$ The resulting equations are

$$D_{\tilde{t}} \tilde{\vec{\theta}} = D_{\tilde{u}} \tilde{\vec{V}} \tag{9}$$

$$\partial_{\tilde{t}} \vec{\pi} = D_{\tilde{u}} \tilde{\vec{\Omega}} \tag{10}$$

$$\alpha D_{\tilde{t}} \tilde{\vec{V}} = D_{\tilde{u}} \tilde{\vec{F}} - \tilde{\vec{V}} + \tilde{\vec{f}} \tag{11}$$

$$\alpha D_{\tilde{t}} \tilde{\vec{L}} = D_{\tilde{u}} \tilde{\vec{M}} + \vec{\theta} \times \tilde{\vec{F}} - \lambda \tilde{\vec{\Omega}} + \tilde{\vec{m}} \tag{12}$$

with two tunable parameters α and λ, as well as the dimensionless moment of inertia $\tilde{I}$.

Let us now assume that we have constitutive laws

$$F_i = g_i\theta_i$$
$$M_i = \epsilon_i\pi_i$$

then in non-dimensionalised form these become

$$\tilde{F}_i = \tilde{g}_i\tilde{\theta}_i$$
$$\tilde{M}_i = \tilde{\epsilon}_i\tilde{\pi}_i$$

where $\tilde{g}_i = g_i/F_0$ and $\tilde{\epsilon}_i = \epsilon_i/(L_0 M_0)$.

For overdamped systems we have $\alpha = 0$, and

$$D_{\tilde{t}}\vec{\tilde{\theta}} = D_{\tilde{u}}\vec{\tilde{V}} \tag{13}$$

$$\partial_{\tilde{t}}\vec{\pi} = D_{\tilde{u}}\vec{\tilde{\Omega}} \tag{14}$$

$$\vec{\tilde{V}} = D_{\tilde{u}}\vec{\tilde{F}} + \vec{\tilde{f}} \tag{15}$$

$$\lambda\vec{\tilde{\Omega}} = D_{\tilde{u}}\vec{\tilde{M}} + \vec{\theta} \times \vec{\tilde{F}} + \vec{\tilde{m}} \tag{16}$$

2 Linearisation of overdamped dynamics

2.1 3D overdamped Cosserat rod

Here we linearise the overdamped equations of motion. We use the non-dimensionalised equations but drop the tildes. Let the forces and moments be of the form

$$\begin{aligned} \vec{F} &= A(\vec{\theta} - \vec{\theta}_0) + B\vec{\pi} \\ \vec{M} &= C(\vec{\theta} - \vec{\theta}_0) + D\vec{\pi} \\ \vec{f} &= P(\vec{\theta} - \vec{\theta}_0) + Q\vec{\pi} \\ \vec{m} &= R(\vec{\theta} - \vec{\theta}_0) + S\vec{\pi} \end{aligned} \tag{17}$$

where $\vec{\theta}_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$. We now linearise with $\vec{\theta} = \vec{\theta}_0 + \delta\vec{\theta}$ and $\vec{\pi} = \delta\vec{\pi}$, such that

$$\vec{F} = A\delta\vec{\theta} + B\delta\vec{\pi}$$
$$\vec{M} = C\delta\vec{\theta} + D\delta\vec{\pi}$$
$$\vec{f} = P\delta\vec{\theta} + Q\delta\vec{\pi}$$
$$\vec{m} = R\delta\vec{\theta} + S\delta\vec{\pi}$$

The linearised equations of motion are then

$$\partial_t\delta\vec{\theta} = H^0\delta\vec{\theta} + H^1\partial_u\delta\vec{\theta} + H^2\partial_u^2\delta\vec{\theta} + J^0\delta\vec{\pi} + J^1\partial_u\delta\vec{\pi} + J^2\partial_u^2\delta\vec{\pi}$$
$$\partial_t\delta\vec{\pi} = K^1\partial_u\delta\vec{\theta} + K^2\partial_u^2\delta\vec{\theta} + L^1\partial_u\delta\vec{\pi} + L^2\partial_u^2\delta\vec{\pi}$$

where

$$H^0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{a_{2,1}}{\lambda} - \frac{p_{3,1}}{\lambda} & -\frac{a_{2,2}}{\lambda} - \frac{p_{3,2}}{\lambda} & -\frac{a_{2,3}}{\lambda} - \frac{p_{3,3}}{\lambda} \\ -\frac{a_{3,1}}{\lambda} + \frac{p_{2,1}}{\lambda} & -\frac{a_{3,2}}{\lambda} + \frac{p_{2,2}}{\lambda} & -\frac{a_{3,3}}{\lambda} + \frac{p_{2,3}}{\lambda} \end{bmatrix}$$

$$H^1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{c_{3,1}}{\lambda} & -\frac{c_{3,2}}{\lambda} & -\frac{c_{3,3}}{\lambda} \\ \frac{c_{2,1}}{\lambda} & \frac{c_{2,2}}{\lambda} & \frac{c_{2,3}}{\lambda} \end{bmatrix}$$

$$H^2 = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

$$J^0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{b_{2,1}}{\lambda} - \frac{q_{3,1}}{\lambda} & -\frac{b_{2,2}}{\lambda} - \frac{q_{3,2}}{\lambda} & -\frac{b_{2,3}}{\lambda} - \frac{q_{3,3}}{\lambda} \\ -\frac{b_{3,1}}{\lambda} + \frac{q_{2,1}}{\lambda} & -\frac{b_{3,2}}{\lambda} + \frac{q_{2,2}}{\lambda} & -\frac{b_{3,3}}{\lambda} + \frac{q_{2,3}}{\lambda} \end{bmatrix}$$

$$J^1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{d_{3,1}}{\lambda} & -\frac{d_{3,2}}{\lambda} & -\frac{d_{3,3}}{\lambda} \\ \frac{d_{2,1}}{\lambda} & \frac{d_{2,2}}{\lambda} & \frac{d_{2,3}}{\lambda} \end{bmatrix}$$

$$J^2 = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}$$

$$K^0 = 0$$

$$K^1 = \begin{bmatrix} \frac{p_{1,1}}{\lambda} & \frac{p_{1,2}}{\lambda} & \frac{p_{1,3}}{\lambda} \\ -\frac{a_{3,1}}{\lambda} + \frac{p_{2,1}}{\lambda} & -\frac{a_{3,2}}{\lambda} + \frac{p_{2,2}}{\lambda} & -\frac{a_{3,3}}{\lambda} + \frac{p_{2,3}}{\lambda} \\ \frac{a_{2,1}}{\lambda} + \frac{p_{3,1}}{\lambda} & \frac{a_{2,2}}{\lambda} + \frac{p_{3,2}}{\lambda} & \frac{a_{2,3}}{\lambda} + \frac{p_{3,3}}{\lambda} \end{bmatrix}$$

$$K^2 = \begin{bmatrix} \frac{c_{1,1}}{\lambda} & \frac{c_{1,2}}{\lambda} & \frac{c_{1,3}}{\lambda} \\ \frac{c_{2,1}}{\lambda} & \frac{c_{2,2}}{\lambda} & \frac{c_{2,3}}{\lambda} \\ \frac{c_{3,1}}{\lambda} & \frac{c_{3,2}}{\lambda} & \frac{c_{3,3}}{\lambda} \end{bmatrix}$$

$$L^0 = 0$$

$$L^1 = \begin{bmatrix} \frac{q_{1,1}}{\lambda} & \frac{q_{1,2}}{\lambda} & \frac{q_{1,3}}{\lambda} \\ -\frac{b_{3,1}}{\lambda} + \frac{q_{2,1}}{\lambda} & -\frac{b_{3,2}}{\lambda} + \frac{q_{2,2}}{\lambda} & -\frac{b_{3,3}}{\lambda} + \frac{q_{2,3}}{\lambda} \\ \frac{b_{2,1}}{\lambda} + \frac{q_{3,1}}{\lambda} & \frac{b_{2,2}}{\lambda} + \frac{q_{3,2}}{\lambda} & \frac{b_{2,3}}{\lambda} + \frac{q_{3,3}}{\lambda} \end{bmatrix}$$

$$L^2 = \begin{bmatrix} \frac{d_{1,1}}{\lambda} & \frac{d_{1,2}}{\lambda} & \frac{d_{1,3}}{\lambda} \\ \frac{d_{2,1}}{\lambda} & \frac{d_{2,2}}{\lambda} & \frac{d_{2,3}}{\lambda} \\ \frac{d_{3,1}}{\lambda} & \frac{d_{3,2}}{\lambda} & \frac{d_{3,3}}{\lambda} \end{bmatrix}$$

We note that the θ equations have 0-th order terms in spatial derivatives, but the π equations do not.

2.2 3D Cosserat rod with typical elastic energy

We now presume that the A and D matrices in Eq. 17 are of the typical diagonal form: $A = \text{diag}\{k, g, g\}$ and $D = \text{diag}\{\eta, \epsilon, \epsilon\}$. Then

$$H^0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{p_{3,1}}{\lambda} & -\frac{g}{\lambda} - \frac{p_{3,2}}{\lambda} & -\frac{p_{3,3}}{\lambda} \\ \frac{p_{2,1}}{\lambda} & \frac{p_{2,2}}{\lambda} & -\frac{g}{\lambda} + \frac{p_{2,3}}{\lambda} \end{bmatrix}$$

$$H^1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{c_{3,1}}{\lambda} & -\frac{c_{3,2}}{\lambda} & -\frac{c_{3,3}}{\lambda} \\ \frac{c_{2,1}}{\lambda} & \frac{c_{2,2}}{\lambda} & \frac{c_{2,3}}{\lambda} \end{bmatrix}$$

$$H^2 = \begin{bmatrix} k & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \end{bmatrix}$$

$$J^0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{b_{2,1}}{\lambda} - \frac{q_{3,1}}{\lambda} & -\frac{b_{2,2}}{\lambda} - \frac{q_{3,2}}{\lambda} & -\frac{b_{2,3}}{\lambda} - \frac{q_{3,3}}{\lambda} \\ -\frac{b_{3,1}}{\lambda} + \frac{q_{2,1}}{\lambda} & -\frac{b_{3,2}}{\lambda} + \frac{q_{2,2}}{\lambda} & -\frac{b_{3,3}}{\lambda} + \frac{q_{2,3}}{\lambda} \end{bmatrix}$$

$$J^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\epsilon}{\lambda} \\ 0 & \frac{\epsilon}{\lambda} & 0 \end{bmatrix}$$

$$J^2 = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{bmatrix}$$

$$K^0 = 0$$

$$K^1 = \begin{bmatrix} \frac{p_{1,1}}{\lambda} & \frac{p_{1,2}}{\lambda} & \frac{p_{1,3}}{\lambda} \\ \frac{p_{2,1}}{\lambda} & \frac{p_{2,2}}{\lambda} & -\frac{g}{\lambda} + \frac{p_{2,3}}{\lambda} \\ \frac{p_{3,1}}{\lambda} & \frac{g}{\lambda} + \frac{p_{3,2}}{\lambda} & \frac{p_{3,3}}{\lambda} \end{bmatrix}$$

$$K^2 = \begin{bmatrix} \frac{c_{1,1}}{\lambda} & \frac{c_{1,2}}{\lambda} & \frac{c_{1,3}}{\lambda} \\ \frac{c_{2,1}}{\lambda} & \frac{c_{2,2}}{\lambda} & \frac{c_{2,3}}{\lambda} \\ \frac{c_{3,1}}{\lambda} & \frac{c_{3,2}}{\lambda} & \frac{c_{3,3}}{\lambda} \end{bmatrix}$$

$$L^0 = 0$$

$$L^1 = \begin{bmatrix} \frac{q_{1,1}}{\lambda} & \frac{q_{1,2}}{\lambda} & \frac{q_{1,3}}{\lambda} \\ -\frac{b_{3,1}}{\lambda} + \frac{q_{2,1}}{\lambda} & -\frac{b_{3,2}}{\lambda} + \frac{q_{2,2}}{\lambda} & -\frac{b_{3,3}}{\lambda} + \frac{q_{2,3}}{\lambda} \\ \frac{b_{2,1}}{\lambda} + \frac{q_{3,1}}{\lambda} & \frac{b_{2,2}}{\lambda} + \frac{q_{3,2}}{\lambda} & \frac{b_{2,3}}{\lambda} + \frac{q_{3,3}}{\lambda} \end{bmatrix}$$

$$L^2 = \begin{bmatrix} \frac{\eta}{\lambda} & 0 & 0 \\ 0 & \frac{\epsilon}{\lambda} & 0 \\ 0 & 0 & \frac{\epsilon}{\lambda} \end{bmatrix}$$

We now also assume that $B = C = 0$ and $\vec{f} = 0$ and that

$$\mathbf{m} = r\boldsymbol{\theta} \times \mathbf{e}_1 - s\boldsymbol{\pi}$$

In the vec notation we have

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & -r & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} -s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -s \end{pmatrix}$$

The linearised dynamics is then

$$H^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{g}{\lambda} & 0 \\ 0 & 0 & -\frac{g}{\lambda} \end{bmatrix}$$

$$H^1 = 0$$

$$H^2 = 0$$

$$J^0 = 0$$

$$J^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\epsilon}{\lambda} \\ 0 & \frac{\epsilon}{\lambda} & 0 \end{bmatrix}$$

$$J^2 = 0$$

$$K^0 = 0$$

$$K^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{g}{\lambda} \\ 0 & \frac{g}{\lambda} & 0 \end{bmatrix}$$

$$K^2 = 0$$

$$L^0 = 0$$

$$L^1 = 0$$

$$L^2 = \begin{bmatrix} \frac{\eta}{\lambda} & 0 & 0 \\ 0 & \frac{\epsilon}{\lambda} & 0 \\ 0 & 0 & \frac{\epsilon}{\lambda} \end{bmatrix}$$

Explicitly the equations of motion are

$$\begin{aligned} \partial_t \delta\theta_1 =& k\partial_u^2 \delta\theta_1 \\ \partial_t \delta\theta_2 =& -\lambda^{-1}\epsilon\partial_u \delta\pi_3 + g\partial_u^2 \delta\theta_2 - \lambda^{-1} g\delta\theta_2 \\ \partial_t \delta\theta_3 =& \lambda^{-1}\epsilon\partial_u \delta\pi_2 + g\partial_u^2 \delta\theta_3 - \lambda^{-1} g\delta\theta_3 \\ \partial_t \delta\pi_1 =& \lambda^{-1}\eta\partial_u^2 \delta\pi_1 \\ \partial_t \delta\pi_2 =& \lambda^{-1}\epsilon\partial_u^2 \delta\pi_2 - \lambda^{-1} g\partial_u \delta\theta_3 \\ \partial_t \delta\pi_3 =& \lambda^{-1}\epsilon\partial_u^2 \delta\pi_3 + \lambda^{-1} g\partial_u \delta\theta_2 \end{aligned} \tag{18}$$

2.3 Planar Cosserat rod

We now assume that $\delta\theta_3 = \delta\pi_1 = \delta\pi_2 = 0$. Let $\vec{x} = \begin{pmatrix} \delta\theta_1 & \delta\theta_2 & \delta\pi_3 \end{pmatrix}^T$, we get

$$\partial_t \vec{x} = H^0 \delta\vec{x} + H^1 \partial_u \delta\vec{x} + H^2 \partial_u^2 \delta\vec{x}$$

where

$$H^0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{a_{2,1}}{\lambda} - \frac{p_{3,1}}{\lambda} & -\frac{a_{2,2}}{\lambda} - \frac{p_{3,2}}{\lambda} & -\frac{b_{2,3}}{\lambda} - \frac{q_{3,3}}{\lambda} \\ 0 & 0 & 0 \end{bmatrix}$$

$$H^1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{c_{3,1}}{\lambda} & -\frac{c_{3,2}}{\lambda} & -\frac{d_{3,3}}{\lambda} \\ \frac{a_{2,1}}{\lambda} + \frac{p_{3,1}}{\lambda} & \frac{a_{2,2}}{\lambda} + \frac{p_{3,2}}{\lambda} & \frac{b_{2,3}}{\lambda} + \frac{q_{3,3}}{\lambda} \end{bmatrix}$$

$$H^2 = \begin{bmatrix} a_{1,1} & a_{1,2} & b_{1,3} \\ a_{2,1} & a_{2,2} & b_{2,3} \\ \frac{c_{3,1}}{\lambda} & \frac{c_{3,2}}{\lambda} & \frac{d_{3,3}}{\lambda} \end{bmatrix}$$

If we assume ordinary hyperelastic forms for the conservative part of the dynamics, with $A = \operatorname{diag}\{k, g, g\}$ and $D = \operatorname{diag}\{\eta, \epsilon, \epsilon\}$, we get

$$H^0 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{p_{3,1}}{\lambda} & -\frac{g}{\lambda} - \frac{p_{3,2}}{\lambda} & -\frac{b_{2,3}}{\lambda} - \frac{q_{3,3}}{\lambda} \\ 0 & 0 & 0 \end{bmatrix}$$

$$H^1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{c_{3,1}}{\lambda} & -\frac{c_{3,2}}{\lambda} & -\frac{\epsilon}{\lambda} \\ \frac{p_{3,1}}{\lambda} & \frac{g}{\lambda} + \frac{p_{3,2}}{\lambda} & \frac{b_{2,3}}{\lambda} + \frac{q_{3,3}}{\lambda} \end{bmatrix}$$

$$H^2 = \begin{bmatrix} k & 0 & b_{1,3} \\ 0 & g & b_{2,3} \\ \frac{c_{3,1}}{\lambda} & \frac{c_{3,2}}{\lambda} & \frac{\epsilon}{\lambda} \end{bmatrix}$$

Finally, we now assume that $B = C = 0$ and $\vec{f} = 0$ and that

$$\mathbf{m} = r\boldsymbol{\theta} \times \mathbf{e}_1 - s\boldsymbol{\pi}$$

In the vec notation we have

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & -r & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} -s & 0 & 0 \\ 0 & -s & 0 \\ 0 & 0 & -s \end{pmatrix}$$

The linearised dynamics is then

$$H^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{g}{\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$H^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\epsilon}{\lambda} \\ 0 & \frac{g}{\lambda} & 0 \end{bmatrix}$$
$$H^2 = \begin{bmatrix} k & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & \frac{\epsilon}{\lambda} \end{bmatrix}$$

The equations can be written out explicitly as

$$\begin{aligned}
\partial_t \delta\theta_1 &= k\partial_u^2 \delta\theta_1 \\
\partial_t \delta\theta_2 &= -\lambda^{-1}\epsilon\partial_u \delta\pi_3 + g\partial_u^2 \delta\theta_2 - \lambda^{-1} g\delta\theta_2 \\
\partial_t \delta\pi_3 &= \lambda^{-1}\epsilon\partial_u^2 \delta\pi_3 + \lambda^{-1} g\partial_u \delta\theta_2
\end{aligned}$$

The first equation is decoupled from the θ_2 and π_3, so we effectively just have a system of two equations. Letting $a = \lambda^{-1}\epsilon$, $b = g$ and $c = \lambda^{-1} g$ then

$$\begin{aligned}
\partial_t \delta\theta_2 &= -a\partial_u \delta\pi_3 + b\partial_u^2 \delta\theta_2 - c\delta\theta_2 \\
\partial_t \delta\pi_3 &= a\partial_u^2 \delta\pi_3 + c\partial_u \delta\theta_2
\end{aligned} \tag{19}$$

The full non-linear equations are (in terms of $\delta\theta$ and $\delta\pi$) are

$$\begin{aligned}
\partial_t \delta\theta_1 =& \frac{\epsilon\xi^2\delta\theta_2 \frac{d}{du}\delta\pi_3}{\lambda} - 2g\xi^2\delta\pi_3\frac{d}{du}\delta\theta_2 - g\xi^2\delta\theta_2\frac{d}{du}\delta\pi_3 \\
&+ \frac{g\xi^3\delta\theta_1\delta\theta_2^2}{\lambda} + \frac{g\xi^2\delta\theta_2^2}{\lambda} - k\xi^3\delta\pi_3^2\delta\theta_1 + k\xi\frac{d^2}{du^2}\delta\theta_1 - \frac{k\xi^3\delta\theta_1\delta\theta_2^2}{\lambda} \\
\partial_t \delta\theta_2 =& -\frac{\epsilon\xi^2\delta\theta_1\frac{d}{du}\delta\pi_3}{\lambda} - \frac{\epsilon\xi\frac{d}{du}\delta\pi_3}{\lambda} - g\xi^3\delta\pi_3^2\delta\theta_2 + g\xi\frac{d^2}{du^2}\delta\theta_2 - \frac{g\xi^3\delta\theta_1^2\delta\theta_2}{\lambda} \\
&- \frac{2g\xi^2\delta\theta_1\delta\theta_2}{\lambda} - \frac{g\xi\delta\theta_2}{\lambda} + 2k\xi^2\delta\pi_3\frac{d}{du}\delta\theta_1 + k\xi^2\delta\theta_1\frac{d}{du}\delta\pi_3 + \frac{k\xi^3\delta\theta_1^2\delta\theta_2}{\lambda} + \frac{k\xi^2\delta\theta_1\delta\theta_2}{\lambda} \\
\partial_t \delta\pi_3 =& \frac{\epsilon\xi\frac{d^2}{du^2}\delta\pi_3}{\lambda} + \frac{g\xi^2\delta\theta_1\frac{d}{du}\delta\theta_2}{\lambda} + \frac{g\xi^2\delta\theta_2\frac{d}{du}\delta\theta_1}{\lambda} \\
&+ \frac{g\xi\frac{d}{du}\delta\theta_2}{\lambda} - \frac{k\xi^2\delta\theta_1\frac{d}{du}\delta\theta_2}{\lambda} - \frac{k\xi^2\delta\theta_2\frac{d}{du}\delta\theta_1}{\lambda}
\end{aligned}$$