Linear stability analysis of overdamped planar Cosserat rod dynamics

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1 Linear stability analysis

The equations of motion are

$$\begin{aligned}\partial_t \delta\theta_2 &= -a\partial_u \delta\pi_3 + g\partial_u^2 \delta\theta_2 + b\delta\theta_2 + c\delta\pi_3 \\ \partial_t \delta\pi_3 &= a\partial_u^2 \delta\pi_3 - b\partial_u \delta\theta_2 - c\partial_u \delta\pi_3\end{aligned} \tag{1}$$

After taking a Fourier transform in the spatial coordinate (and dropping δ) we get

$$\partial_t \begin{pmatrix} \theta_2(k) \\ \pi_3(k) \end{pmatrix} = A \begin{pmatrix} \theta_2(k) \\ \pi_3(k) \end{pmatrix}$$

where

$$A = \begin{pmatrix} b - gk^2 & -iak + c \\ -ibk & -ak^2 - ick \end{pmatrix}$$

These equations of motion are a result of a Cosserat rod with ordinary hyperelastic potential energy (the one described in the PRE right now), with an additional moment-per-unit-material-length

$$\mathbf{m} = r\boldsymbol{\theta} \times \mathbf{e}_1 - s\boldsymbol{\pi}$$

The intutition behind the parameters are as follow:

- $a = \lambda^{-1}\epsilon$: ϵ is the elastic spring constant of the constitutive moment (so the $\epsilon\pi_3^2$ term in the potential), λ is the magnitude of the rotational friction relative to the translation friction.
- g: The elastic spring constant of the constitutive force (the $g\theta_2^2$ term in the potential).
- $b = \lambda^{-1}(r - g)$: r is the strength of the moment-per-unit-material-length due to θ. So if b is positive, then the active moment is larger than the restoring moment from to the constitutive dynamics.

- $c = \lambda^{-1}s$: s is the strength of the moment-per-unit-material-length due to π.

Let the eigenvalues of A be u_1 and u_2. We will explore the behaviour of these as a function of the parameters.

We have

$$\begin{aligned}
u_1 &= -\frac{ak^2}{2} + \frac{b}{2} - \frac{ick}{2} - \frac{gk^2}{2} - \frac{\sqrt{a^2k^4 - 2abk^2 + 2iack^3 - 2agk^4 + b^2 - 2ibck - 2bgk^2 - c^2k^2 - 2icgk^3 + g^2k^4}}{2} \\
u_2 &= -\frac{ak^2}{2} + \frac{b}{2} - \frac{ick}{2} - \frac{gk^2}{2} + \frac{\sqrt{a^2k^4 - 2abk^2 + 2iack^3 - 2agk^4 + b^2 - 2ibck - 2bgk^2 - c^2k^2 - 2icgk^3 + g^2k^4}}{2}
\end{aligned} \tag{2}$$

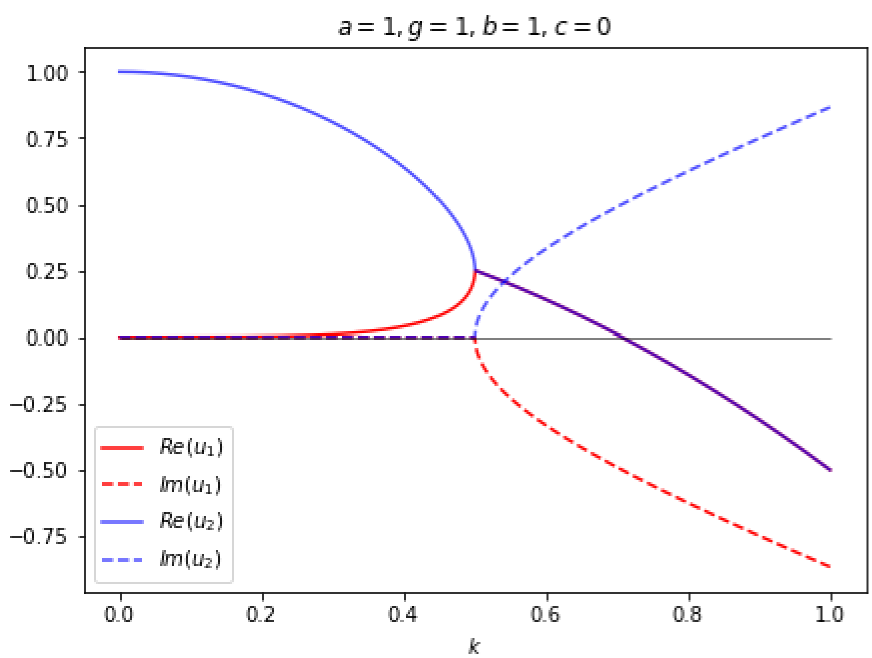
1.1 $c = 0$

We will first consider the situation where $c = 0$, in other words

$$\mathbf{m} = r\boldsymbol{\theta} \times \mathbf{e}_1$$

1.2 $a = g$, $b > 0$, $c = 0$

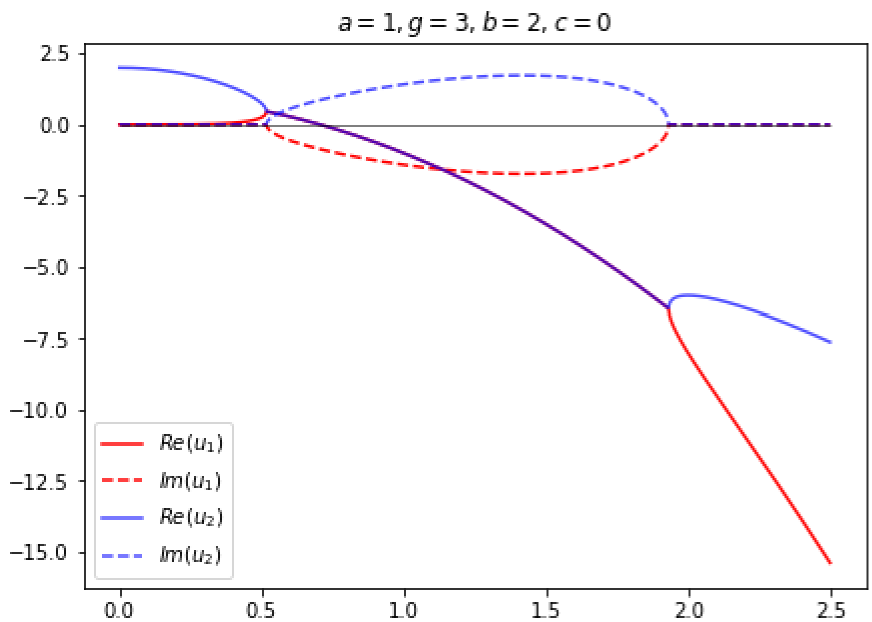
Let $a = g$. As we can rescale all parameters without changing the qualitative behaviour of the system, we can set $a = g = 1$. For any positive value of b, we find qualitatively an eigenstructure of the form:



The solid lines are the real components of the eigenvalues, and the dashed lines the imaginary ones. We see that there is a banded region of modes (starting from $k = 0$ and ending at around $k = 0.7$) that are unstable. Furthermore, a subset of the growing modes (from around $k = 0.5$) are also oscillatory. At high wavenumbers the imaginary components increase in magnitude indefinitely.

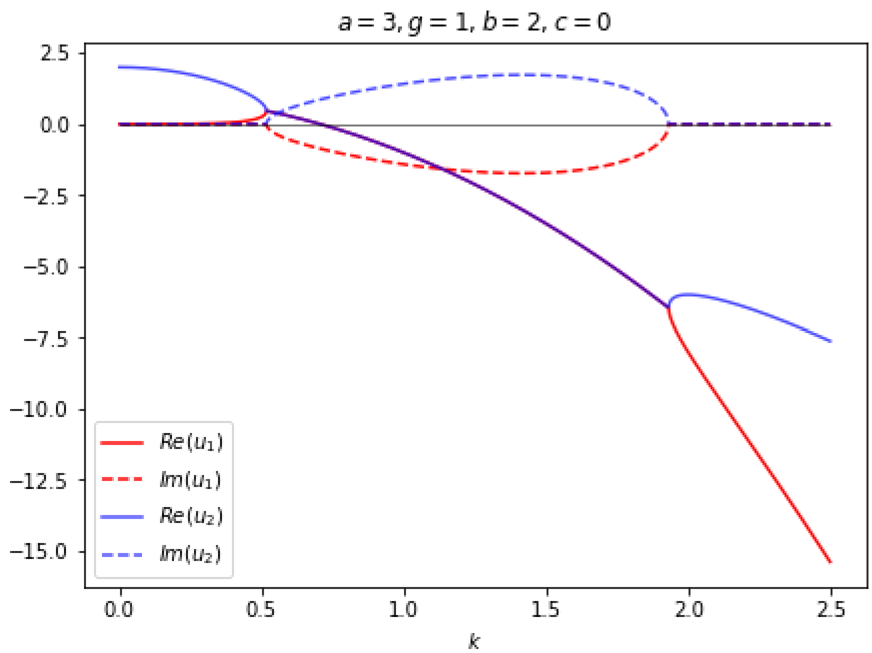
1.3 $a \neq g$, $b > 0$, $c = 0$

We now let $a \neq g$, meaning that the constitutive forces and moments do not have the same "magnitude", and keep $b > 0$.



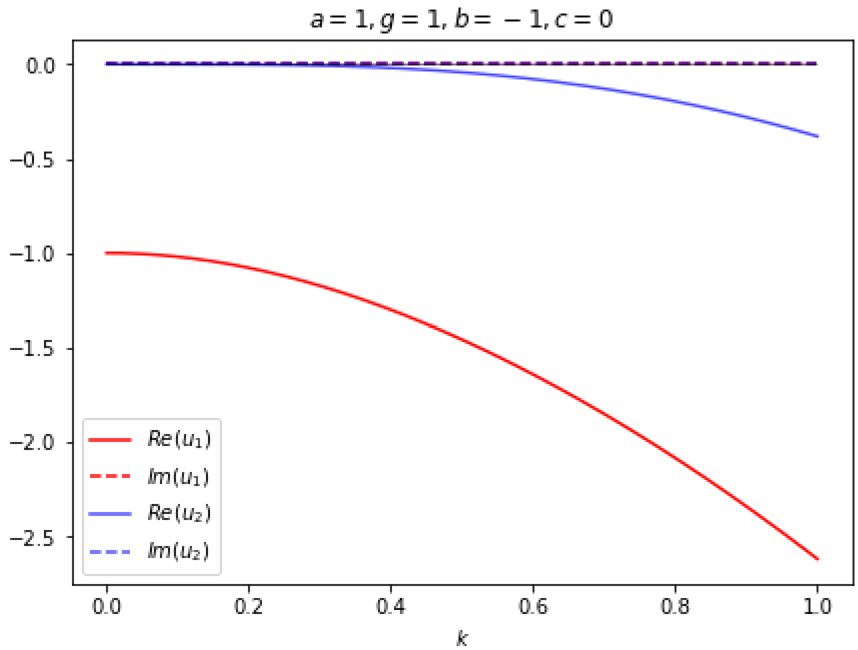
This is essentially the same qualitative behaviour as in the previous case. With the exception that at around $k = 2$, the oscillations stop and the eigenvalues become real. So when $a \neq g$ we see that the oscillations are banded as well. Physically, $a \neq g$ would probably be the more physically realistic scenario.

Interestingly, the qualitative picture does not change at all depending on whether $a > g$ or $g < a$.



1.4 $b = 0$, $c \neq 0$

If we now let $b < 0$, meaning that the active moment is smaller than the constitutive moment, then we have:



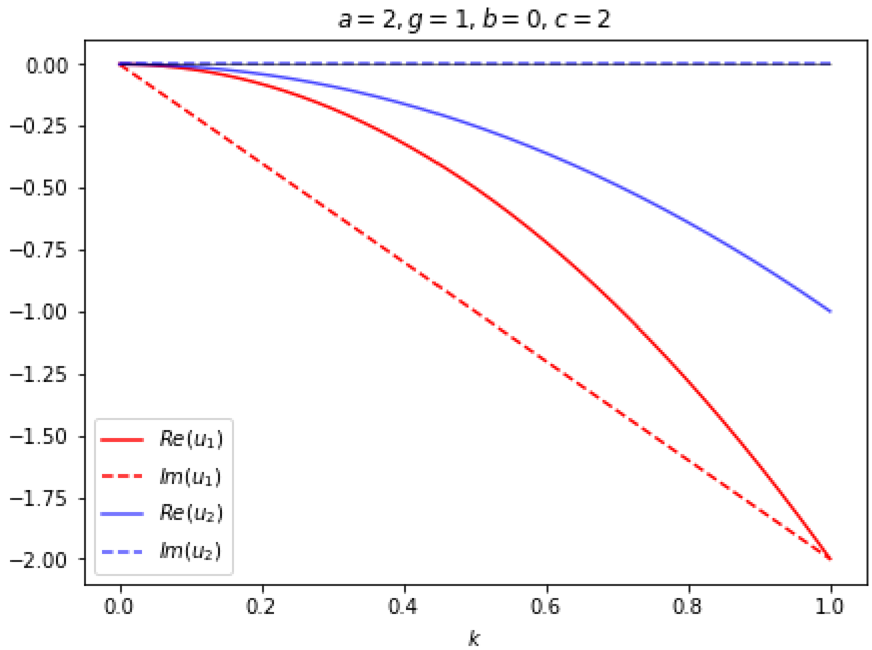
No instabilities, and no oscillatory modes. This makes sense physically, as in this scenario the constitutive moments dominate the active ones. In other words, **the linear instability only kicks in at a finite amplitude of the active parameter**.

1.5 $b = 0$, $c \neq 0$

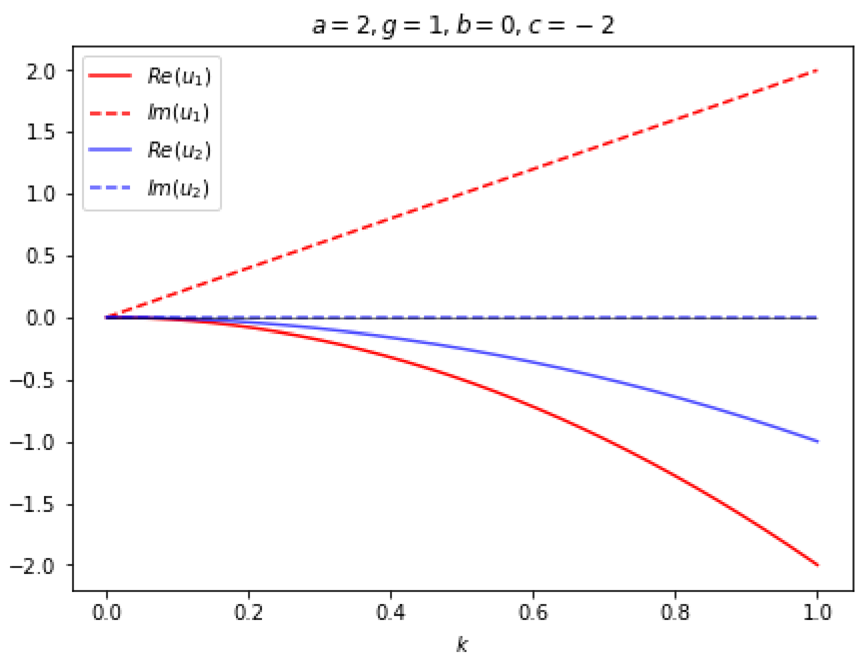
We now consider

$$\mathbf{m} = -s\boldsymbol{\pi} \tag{3}$$

For $a > g$ and $c > 0$ we have



For $c < 0$ we have



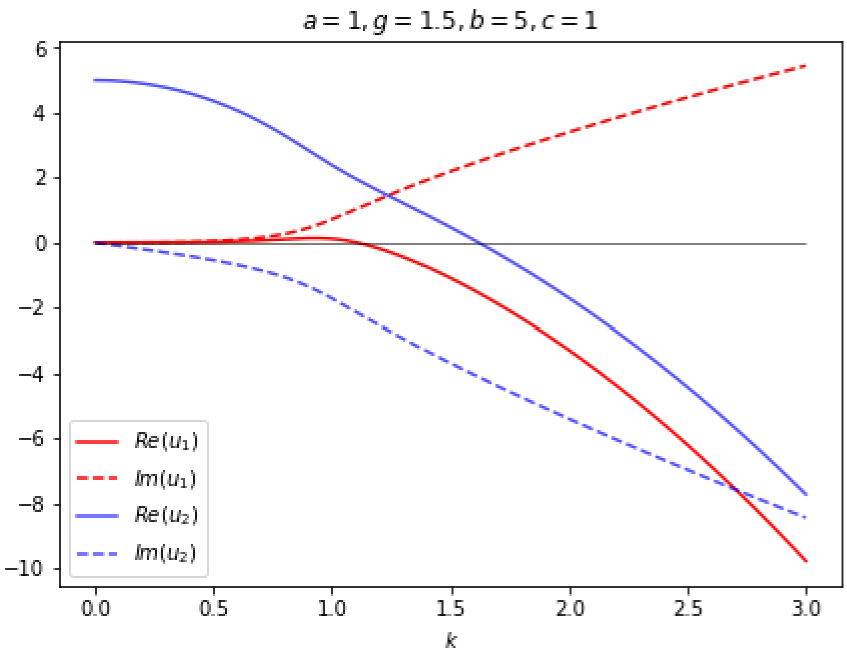
The qualitative picture does not change much for various choices of a and g. We see that Eq. 3 does not lead to any linear instabilities.

1.6 $b \neq 0,\ c \neq 0$

Now let

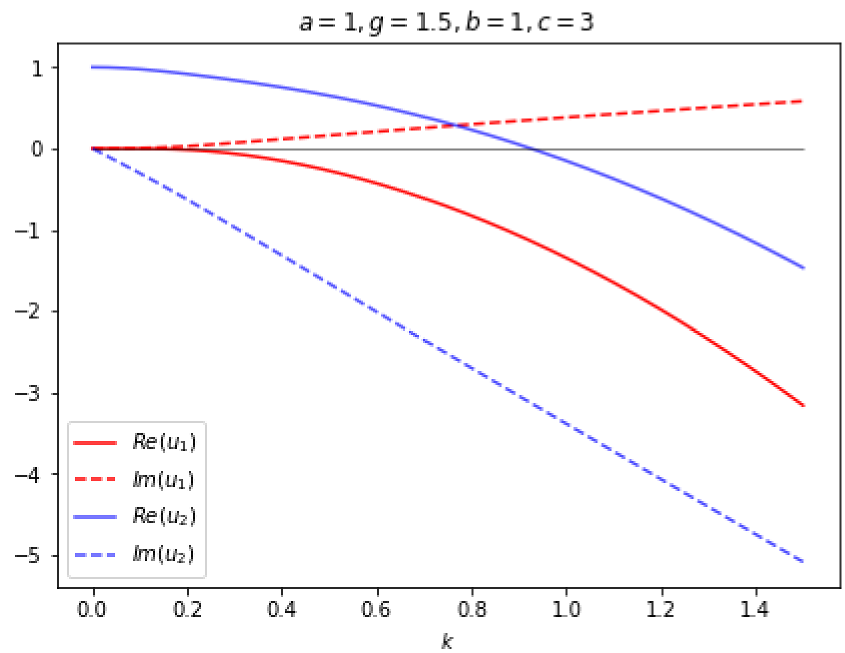
$$\mathbf{m} = r\boldsymbol{\theta} \times \mathbf{e}_1 - s\boldsymbol{\pi}$$

Consider $a \neq g$ and $b > c$. We have



We see that we get long wave-length instabilities that dampen out at around $k = 1.7$. The $-s\boldsymbol{\pi}$ term ensure that oscillations persist for high wave-numbers.

If $b < c$ we have


$a = 1, g = 1.5, b = 1, c = 3$
1
0
−1
−2
−3
−4
−5
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
k
$Re(u_1)$
$Im(u_1)$
$Re(u_2)$
$Im(u_2)$